# Mathematical essentials

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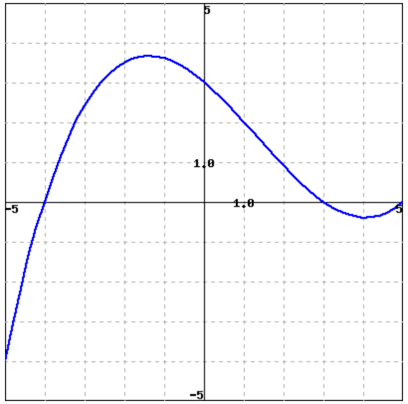
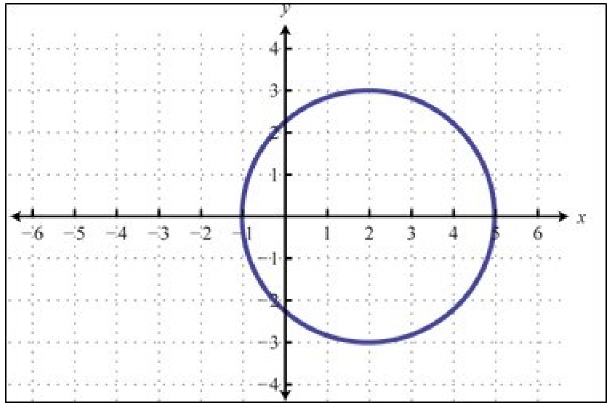
# Overview

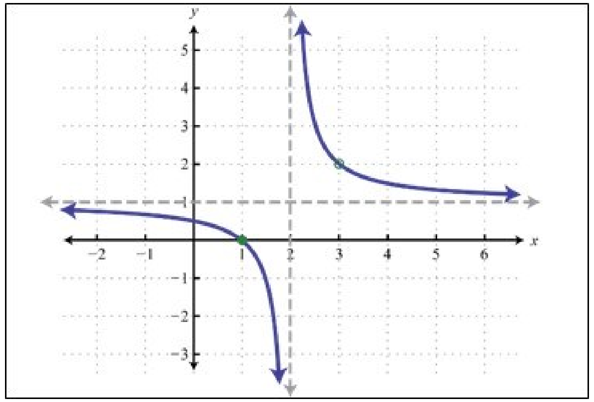
- Functions
- Differentiation & Integration
- Differential equations
- Probability and distributions (see Thursday lecture)

# What is a mathematical function?

- A function is a relation between a set of inputs $x \in Domain\ D$ and a set of outputs $y = f(x) \in Range\ R$ with the property that each input is related to exactly one output







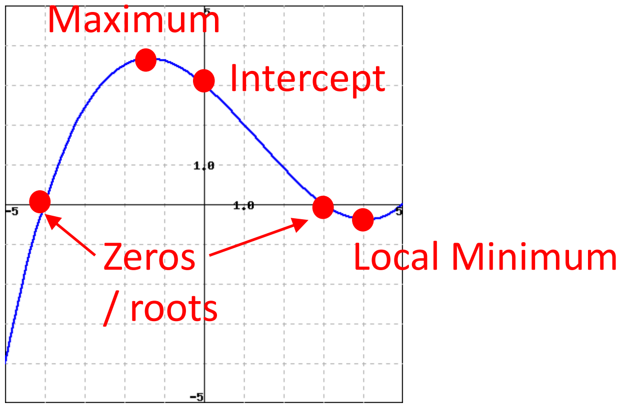
Which one of these graphs is NOT the graph of a function?

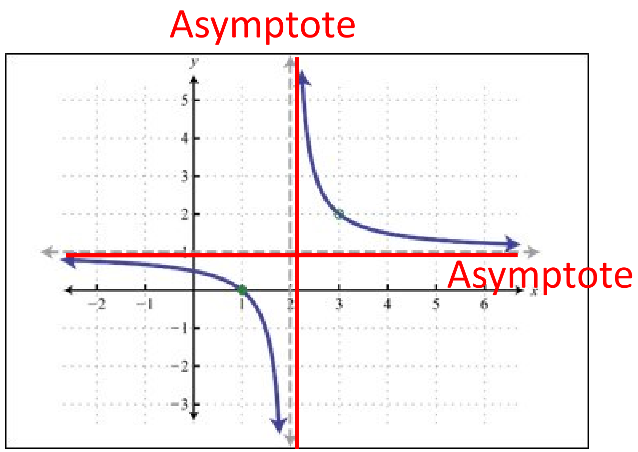
# What is a mathematical function?

- Functions can consist of *variables* and *parameters*
  - *Variable: trait that vary over time or space*
  - *Parameters: assumed to be constant*

Example: $y = \mathrm{m\,x + c}$ has independent variable x, dependent variable y and parameters m & c

- Properties of functions can be summarized by their graphs



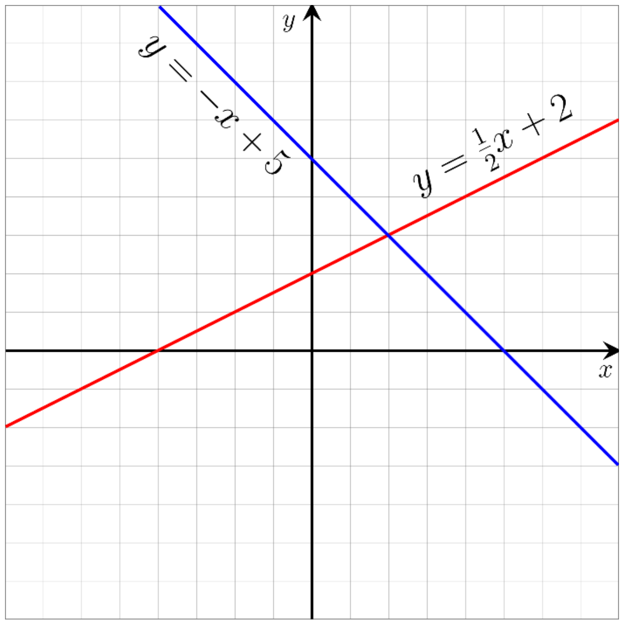



# Linear functions $f(x) = \mathrm{m\,x + c}$

Fully defined by

- constant slope m
  - For every increase of one in the value of x, the value of y increases by m
- constant intercept c
  - When x = 0 then f(x) = c

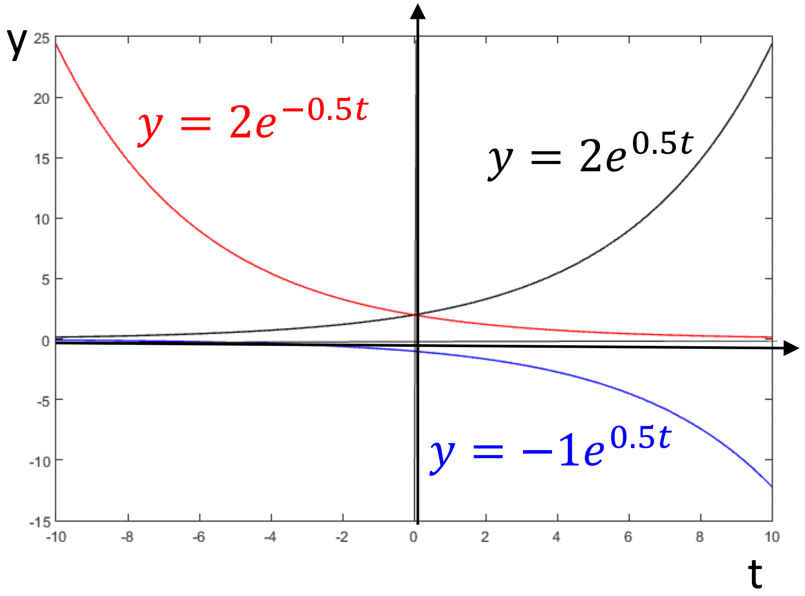
- What are the values for the slopes & intercepts of these functions?
- What properties do these functions have?



# Exponential functions

$$y = f(t) = ae^{bt} = a\exp(bt)$$

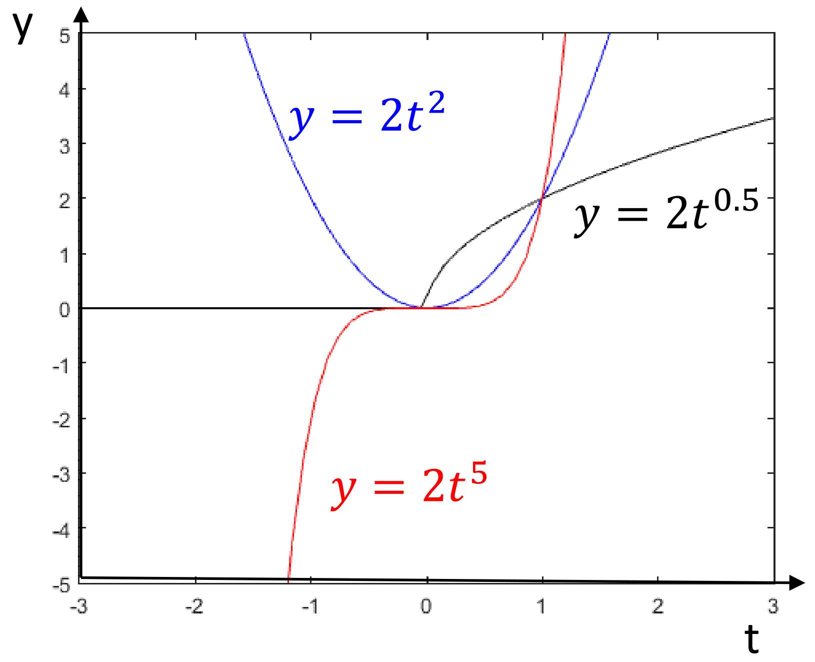
- Where the parameters a and b are constants
  - a is the intercept
  - b controls the slope
- e is the Euler number (≈ 2.718)
  - Exponential has no roots, i.e. $e^x > 0$ for all x
  - $e^0 > 1$
- b controls the behaviour for large x
  - b>0 means that $f(t) \rightarrow \pm\infty$ as t → ∞
  - b<0 means that $f(t) \rightarrow 0$ as t → ∞



# Power functions

$$y = f(t) = at^b$$

- The exponent b determines the domain & the shape of the function
  - 0<b<1: f(t) defined for t$\geq 0$
  - b $\geq$ 1: f(t) defined for all real numbers t



# Derivative of a function

**1. Mathematical definition:**

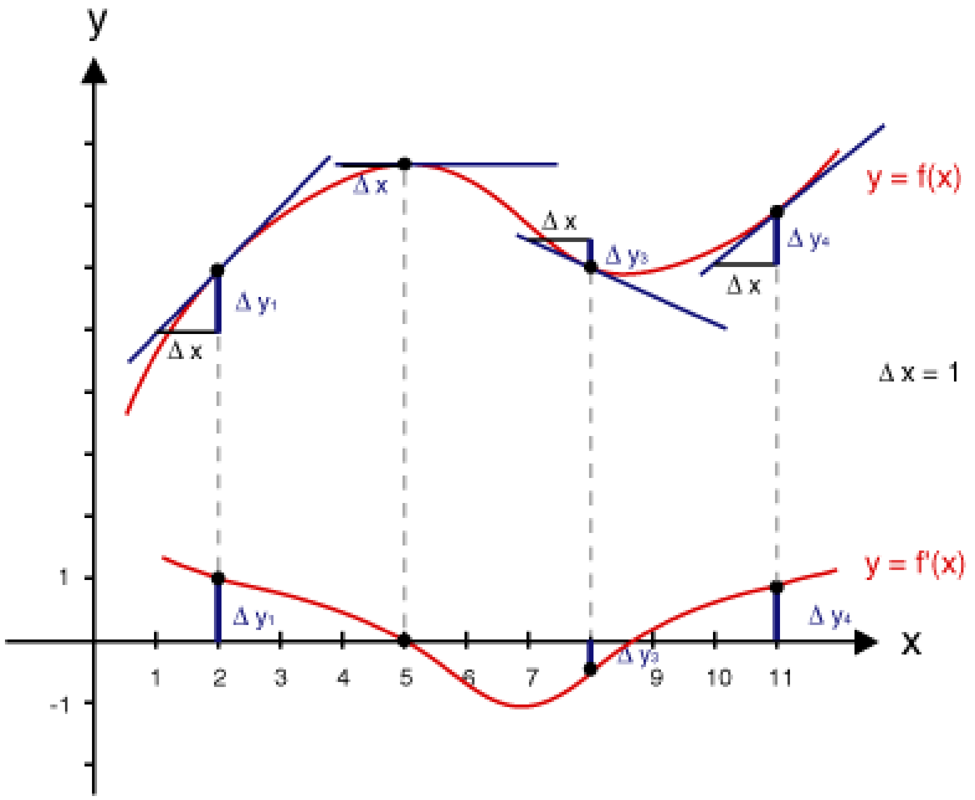
$$\frac{dy}{dt}(t_0) = \lim_{x \to x_0} \frac{y(t) - y(t_0)}{t - t_0} \approx \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}$$

**2. Geometric definition:**

The derivative $\frac{dy}{dt}(t_0)$ of a function at point $t = t_0$ is the slope of the tangent line to the graph at $t_0$

**3. Physical definition:**

The derivative $\frac{dy}{dt}$ defines the average rate of change of variable y with respect to $t$




*Derivatives allow you to find the maxima / minima of a function:*

*Find* $t_0$ *so that* $\frac{dy}{dt}(t_0) = 0$

# Examples of derivatives of basic functions

- Differentiation is the process of calculating derivatives
- Derivatives of certain functions need to be committed to memory:
  - If y = c then $\frac{dy}{dt} = 0$ for constant c
  - If $y = e^{bt+c}$ then $\frac{dy}{dt} = be^{bt+c}$
  - If $y = t^b$ then $\frac{dy}{dt} = bt^{b-1}$
- If you know these, and some basic rules, you can calculate the derivative of almost any function

# Rules of differentiation

1. Derivatives of sums and constant multiples

- If $y(t) = y_1(t) \pm\ y_2(t)$ then $\frac{dy}{dt} = \frac{dy_1}{dt} \pm \frac{dy_2}{dt}$
- If $y(t) = kx(t)$ then $\frac{dy}{dt} =$ k $\frac{dx}{dt}$

Example: Calculate the derivative of $y = 2e^{3t} + t^5$

# Rules of differentiation

2. Derivatives of products

If $y(t) = u(t) * v(t)$ then $\frac{dy}{dt} = u\frac{dv}{dt} + v\frac{du}{dt}$

Example: $y = 5t^3e^{-2t}$

3. Derivatives of quotients

If $y(t) = \frac{u(t)}{v(t)}$ then $\frac{dy}{dt} = \frac{v\frac{du}{dt} - u\frac{dv}{dt}}{v^2}$

Example: $y = \frac{5t^3}{e^{2t}}$

4. Chain rule

If y(x) = y(x(t)) then $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$

Example: $y = e^{e^{-t}}$

# Integration

- Reverse process of differentiation

  Example: if $\frac{dy}{dt} = 2t$ then $y(t) = t^2$

  Write the reverse as:

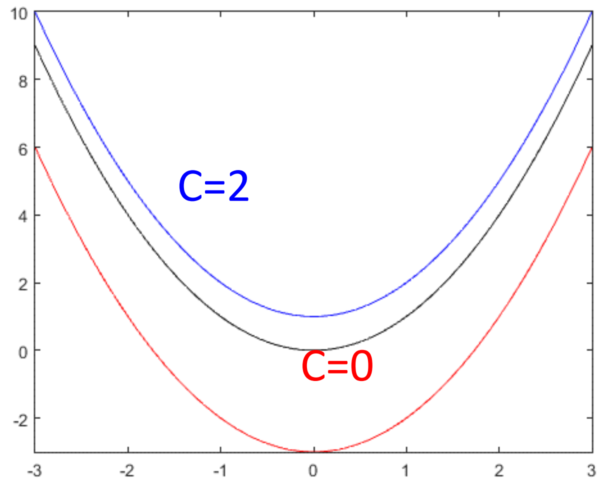
$$\int 2t\,dt = t^2$$

- $t^2$ is called the **indefinite integral** of 2t
- But why do books have

$$\int 2t\,dt = t^2 + C?$$

# Constant of integration

- While differentiating any function gives a well defined answer, a number of functions have the same differential
  - E.g. each $y = t^2, y = t^2 + 1, y = t^2 - 3$ differentiate to 2t
- The differential tells you the slope but nothing about the y-location
- The indefinite integral of a function is a set of parallel curves
- The constant of integration C is determined by the initial conditions: y(t=0)=C




$$\int 2t\,dt = t^2 + C$$

# Definite integrals

If F(t) is the indefinite integral of a function f, i.e. $\int f(t)dt = F(t) + C$

Then the definite integral of f with limits a and b is

$$\int_a^b f(t)dt = F(b) - F(a)$$

Example: $\int_1^2 2t\, dt = [t^2]_1^2 = 2^2 - 1^2 = 4 - 1 = 3$

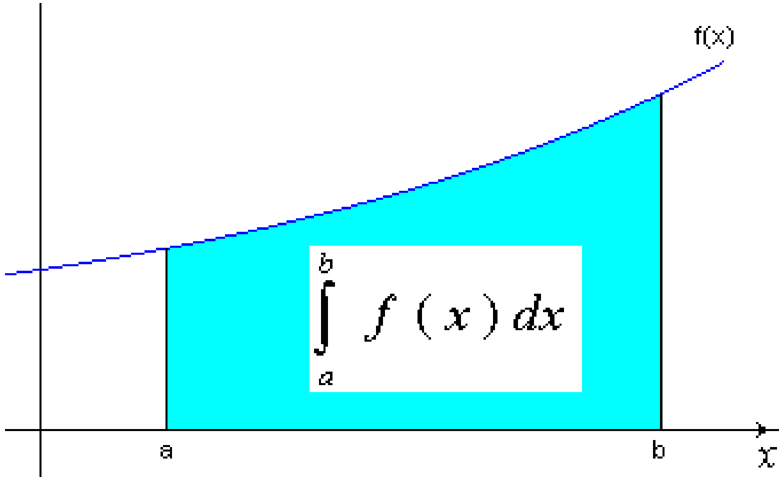
There are several integration rules and techniques, but often calculation of integrals requires numerical approximations (i.e. computer software)

# Definite integrals and Area under the curve

Theorem:

If f(x) is a continuous function that is non-negative for a ≤ x ≤ b, then the area of the region bounded by the graph of y = f(x), the x-axis and the lines x=a and x=b is given by

$$\int_a^b f(x)dx = F(b) - F(a)$$



# Differential equations - Outline

- What are differential equations
- Why do we care about them?
- The modelling process
- Some examples
- How to solve them

# What are differential equations?

Crude definition: Differential Equation = equations with derivatives

Mathematical definition:

**First order differential equation: Equation for the rate of change of a dependent variable as a function of independent variables, itself and model parameters**

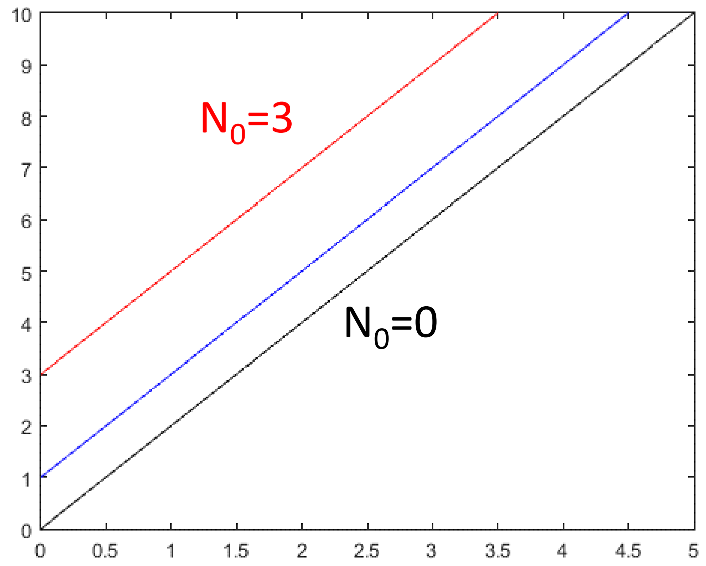
$$\frac{dN}{dt} = f(N, t, \{a, b, c\})$$

# Examples for first order linear equations

Let N(t) = size of a population at time t

**Migration**: every year the population increases by 2:

- Differential equation: $$\frac{dN}{dt} = 2$$
- Solution: $N(t) = 2t + C$
- The constant C is defined by the initial state of the system:
  - C = $N(0) = N_0$
- **The exact solution of a differential equation requires specification of initial conditions ($N_0$)**

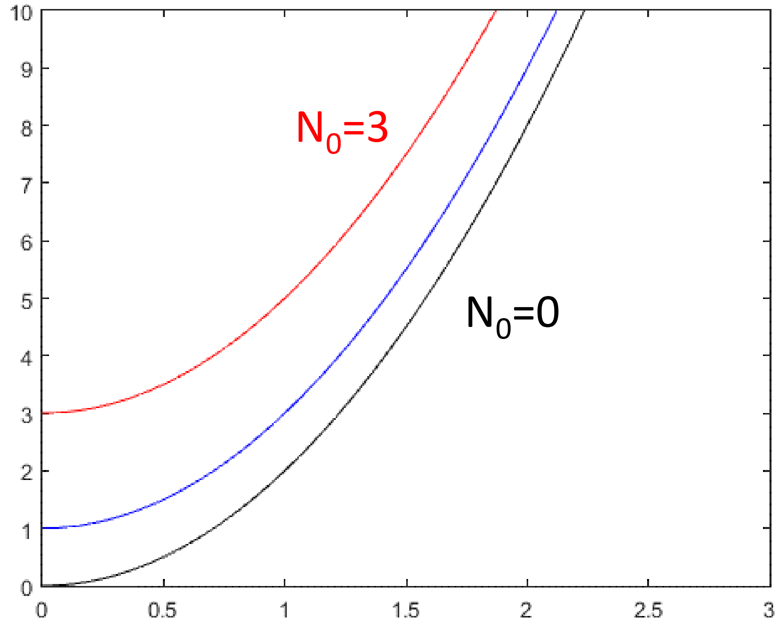


# Examples for first order linear equations

Let N(t) = size of a population at time t

**Time-depdendent growth**: every year the population increases by 2t:

- Differential equation: $$\frac{dN}{dt} = 2t$$
- Solution: $N(t) = t^2 + N_0$



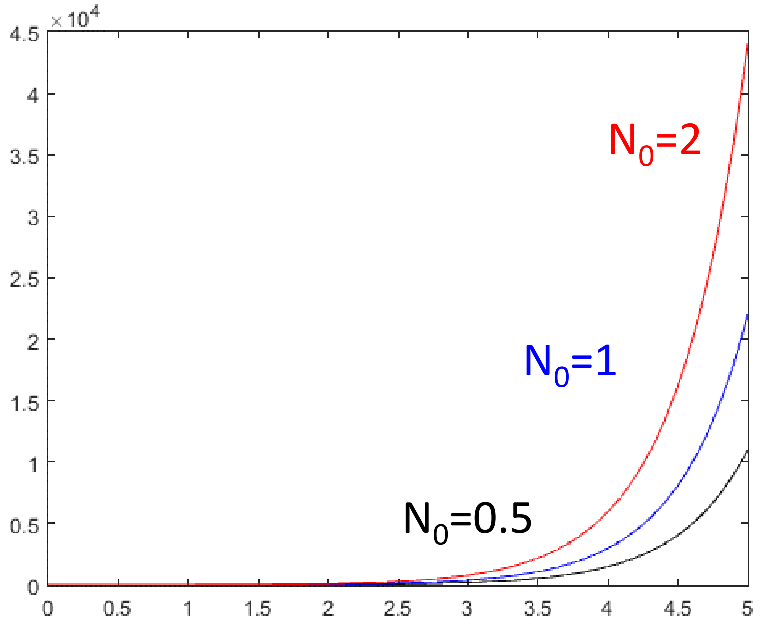
# Examples for first order linear equations

Let N(t) = size of a population at time t

**Exponential growth**: every year the population doubles in size

- Differential equation: $$\frac{dN}{dt} = 2N$$

- Solution: $N(t) = e^{2t+C} = e^{C}e^{2t} = N_0 e^{2t}$

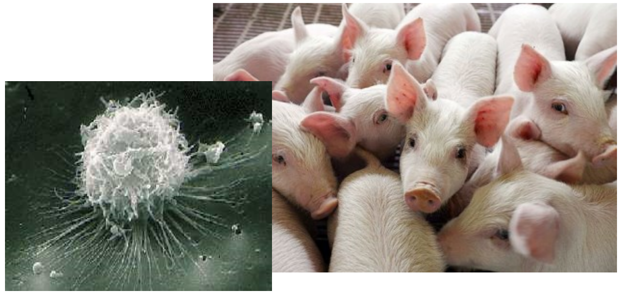


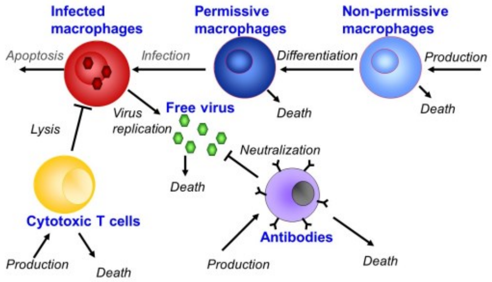
# Why do we care about differential equations?

- They are the mathematical representation of a dynamical systems:
  - E.g. growth or decay of populations
  - Change of pathogen burden over time
  - Change of the number of infected individuals over time or space

# Modelling process

1. Reality → Abstract model → Mathematical model: Differential equations (DE)





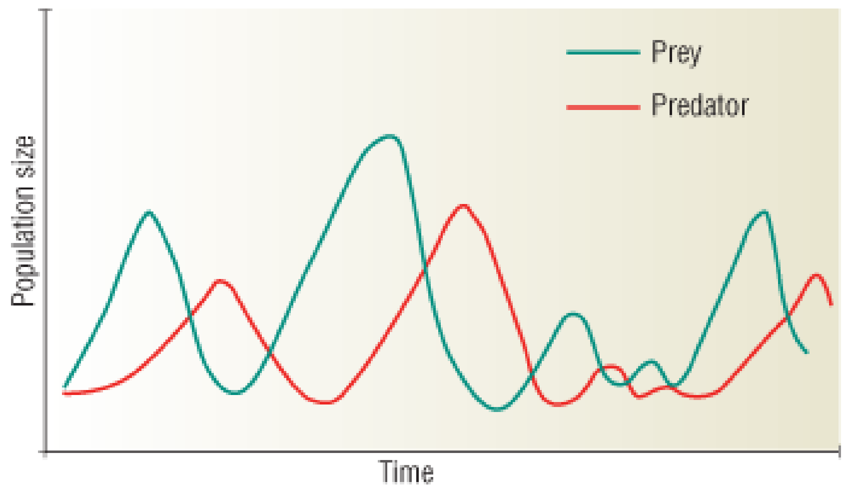

$$\frac{dx}{dt} = \alpha x - \beta xy$$

$$\frac{dy}{dt} = \delta xy - \gamma y$$

2. Learn about reality ← Learn about abstract model ← Examine mathematical model: solve DEs & explore solutions

# Systems of differential equations – modelling interacting populations

- The growth / decline of one species often depends on the population level of another species
- Example: Preditor – Prey dynamics



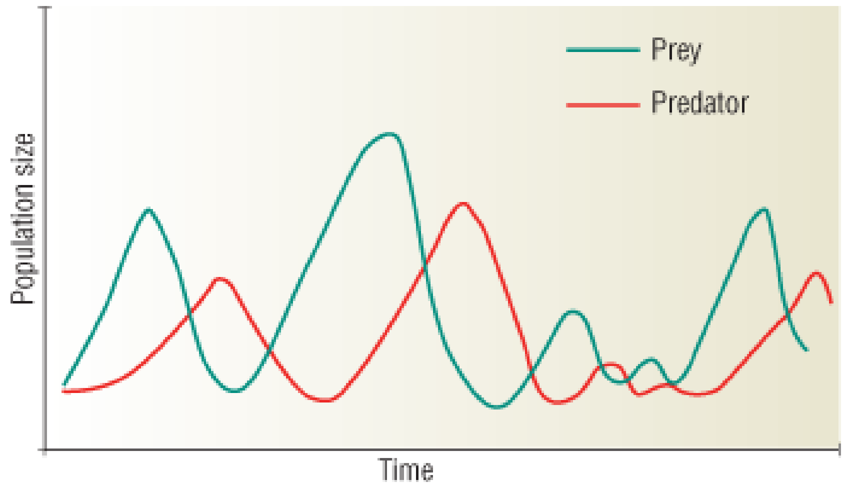

# Preditor – Prey models

- System of 'coupled' differential equations:

Preditor: $$\frac{dN}{dt} = \mathrm{f}(\mathrm{N}, \mathrm{P}, \mathrm{t})$$

Prey: $$\frac{dP}{dt} = \mathrm{g}(\mathrm{N}, \mathrm{P}, \mathrm{t})$$

How many variables does this model have?



# The Lotka-Volterra Predator –Prey model

$$\frac{dx}{dt} = \alpha x - \beta xy$$

$$\frac{dy}{dt} = \delta xy - \gamma y$$

First-order, non-linear differential equation model with

- x = number of prey (rabbits)
- y = number of preditors (foxes)
- α,β,δ,γ are positive real parameters describing the interactions between the two species

What is the interpretation of these model equations?

# The Lotka-Volterra Predator –Prey model

$$\frac{dx}{dt} = \alpha x \; - \; \beta xy$$

$$\frac{dy}{dt} = \delta xy \; - \; \gamma y$$

x = number of prey (rabbits)
y = number of preditors (foxes)
α,β,δ,γ are positive real parameters describing the interactions between the two species

Interpretation:

- Prey reproduce exponentially unless subject to predation (term $\alpha x$)
- The rate of predation is proportional to the rate at which preditors and prey meet ($\beta xy$)
- The growth rate of the predators is proportional to the rate at which they catch prey ($\delta xy$
- In the absence of prey, predators vanish exponentially ($\gamma y$)

# Examining the predator-prey model:

$$\frac{dx}{dt} = \alpha x - \beta xy$$

$$\frac{dy}{dt} = \delta xy - \gamma y$$

- The Lotka-Volterra model is one of the most studied mathematical models in biology
  - One of the rare cases where analytical solutions exist
  - Lends itself to various mathematical techniques
- Here we use it to demonstrate the process of model exploration adopted in this course

For more info, see e.g. Wangersky, Peter J. "Lotka-Volterra population models." *Annual Review of Ecology and Systematics* 9 (1978): 189-218.

# Examining the predator-prey model:
# 1. Dynamic behaviour

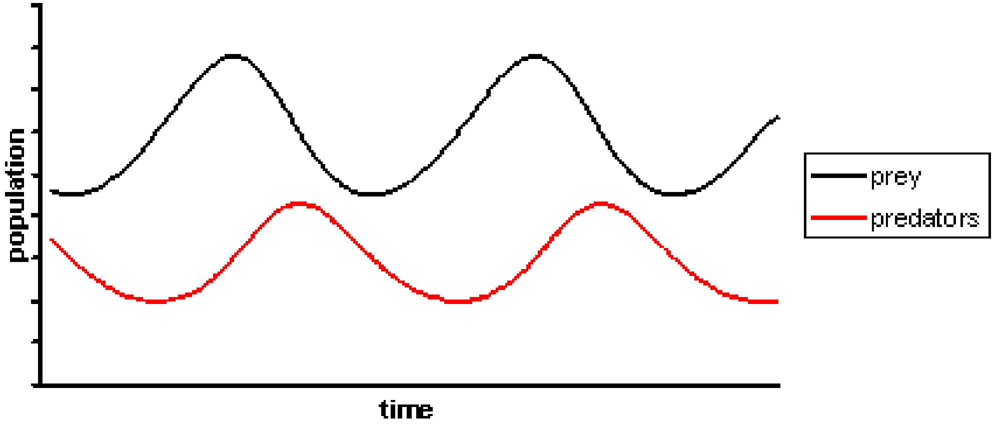
$$\frac{dx}{dt} = \alpha x - \beta xy$$
$$\frac{dy}{dt} = \delta xy - \gamma y$$

In order to obtain numerical solutions to the Lotka-Volterra model:

1. Choose arbitrary values for the model parameters
2. Define initial conditions x(t=0), y(t=0)
3. Code the differential equations
4. Call a numerical solver (e.g. ‘lsoda’ in R) to generate predictions for x(t) and y(t) for specific values of t
5. Plot profiles
6. Interpret the results

We will follow this process in the tutorials

# Dynamics of the preditor-prey system: Frequency plot




- Predators thrive when there is plentiful pray but ultimately run out of food supply & decline.
- As the predator population gets low, the prey population increases.
- These dynamics continue in a periodic cycle of growth and decline

# Examining the predator-prey model: 2. Population Equilibrium

$$\frac{dx}{dt} = \alpha x - \beta xy$$

$$\frac{dy}{dt} = \delta xy - \gamma y$$

Population equilibrium (also called steady state) occurs when neither the number of predators (y) or prey (x) are changing:

- i.e. when $\frac{dx}{dt} = \frac{dy}{dt} = 0$

- Setting the above equations to zero yields 2 equilibria:
  1. $\{x = 0, y = 0\}$ - Extinction of both species
  2. $\left\{x = \frac{\gamma}{\delta} 0, y = \frac{\alpha}{\beta}\right\}$ - Both populations sustain their current numbers

We will follow this process in the tutorials