Matrix Algebra Basics

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Definition

- **Matrix:** an array of numbers, symbols, or expressions

$$A_{(3\times5)} = \begin{pmatrix} -2 & 0 & 1/3 & 0.25 & \mu \\ 2^x & \sqrt{-1} & \pi & 113 & \int f(x)dx \\ \exp(a) & \Delta & \sum_i w_i & 3\times\beta & \Theta \end{pmatrix}$$

Matrix of dimension (3 x 5), i.e. 3 rows by 5 columns

Matrix ≈ Closet

General Form

$$A_{(r\times c)} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1c} \\ a_{21} & a_{22} & \cdots & a_{2c} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rc} \end{pmatrix}$$

In general, matrices are indicated by capital letters, and their elements indicated by small letters, as a_{ij} (i = 1, 2,...,r and j = 1, 2,...,c)

row ↙ ↘ column

Examples

$$A_{(3\times 3)} = \begin{pmatrix} 3 & -1 & 0 \\ 2 & 2 & 5 \\ \pi & 1/3 & 10 \end{pmatrix} \qquad a_{2,3} = 5$$

$$B_{(3\times 2)} = \begin{pmatrix} 0 & -1 \\ 3 & -1 \\ 2 & 5 \end{pmatrix} \qquad b_{3,1} = 2$$

- row vector:

$$x_{(1\times 3)} = \begin{pmatrix} 1 & 0 & 2 \end{pmatrix}$$

- column vector:

$$y_{(2\times 1)} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

Some Special Matrices

Square matrix: $\mathbf{A} = \begin{pmatrix} 3 & -1 & 0 \\ 2 & 2 & 5 \\ \pi & 1/3 & 10 \end{pmatrix}$ $(r = c)$

Symmetric matrix: $\mathbf{S} = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & 5 \\ 0 & 5 & 10 \end{pmatrix}$ $(s_{ij} = s_{ji})$

Upper triangular matrix: $\mathbf{T} = \begin{pmatrix} 3 & -1 & 2 \\ 0 & 2 & 5 \\ 0 & 0 & 10 \end{pmatrix}$ $(t_{ij} = 0 \text{ if } i > j)$
(Lower)

Diagonal matrix: $\mathbf{D} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ $(d_{ij} = 0 \text{ if } i \neq j)$

Some Special Matrices

Unit matrix: $\mathbf{J}=\begin{pmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix}$ $(j_{ij}=1)$

Zero matrix: $\mathbf{N}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$ $(n_{ij}=0)$

Identity matrix: $\mathbf{I}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$ $\begin{cases}i_{ij}=1 \text{ if } i=j\\ i_{ij}=0 \text{ if } i\neq j\end{cases}$

And many others...

Matrix Partition

$$\mathbf{K}=\begin{pmatrix}3&1&1\\2&0&1\\3&5&0\end{pmatrix}=\left(\begin{array}{c|cc}3&1&1\\\hline 2&0&1\\3&5&0\end{array}\right)=\begin{pmatrix}\mathbf{a}&\mathbf{b}\\\mathbf{d}&\mathbf{B}\end{pmatrix}$$

$$\mathbf{a}=(3),\ \mathbf{b}=\begin{pmatrix}1&1\end{pmatrix},\ \mathbf{d}=\begin{pmatrix}2\\3\end{pmatrix},\ \mathbf{B}=\begin{pmatrix}0&1\\5&0\end{pmatrix}$$

$$\mathbf{K}=\begin{pmatrix}\mathbf{r}_1\\\mathbf{r}_2\\\mathbf{r}_3\end{pmatrix}\quad \mathbf{r}_1=\begin{pmatrix}3&1&1\end{pmatrix},\ \mathbf{r}_2=\begin{pmatrix}2&0&1\end{pmatrix},\ \mathbf{r}_3=\begin{pmatrix}3&5&0\end{pmatrix}$$

$$\mathbf{K}=\begin{pmatrix}\mathbf{c}_1&\mathbf{c}_2&\mathbf{c}_3\end{pmatrix}$$

$$\mathbf{c}_1=\begin{pmatrix}3\\2\\3\end{pmatrix},\ \mathbf{c}_2=\begin{pmatrix}1\\0\\5\end{pmatrix},\ \mathbf{c}_3=\begin{pmatrix}1\\1\\0\end{pmatrix}$$

Addition/Subtraction

$$\mathbf{D} = \begin{pmatrix} 3 & 0 \\ 1 & 2 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{E} = \begin{pmatrix} -1 & 2 \\ -1 & 3 \\ 0 & 3 \end{pmatrix}$$

$$\mathrm{K}_{(r\times c)} = \mathrm{D}_{(r\times c)} \pm \mathrm{E}_{(r\times c)} \rightarrow k_{ij} = d_{ij} \pm e_{ij}$$

$$\mathbf{K} = \mathbf{D} + \mathbf{E} = \begin{pmatrix} 3-1 & 0+2 \\ 1-1 & 2+3 \\ 1+0 & -1+3 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 0 & 5 \\ 1 & 2 \end{pmatrix}$$

Properties: $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$ and $(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$

Multiplication

$$\mathbf{A}_{(r\times p)} \times \mathbf{B}_{(p\times c)} = \mathbf{C}_{(r\times c)}$$

$$c_{ij} = \begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{ip} \end{pmatrix} \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{pj} \end{pmatrix} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

2 x 1 + 1 x (-1) + 0 x 0 = 1

$${}_2\begin{pmatrix} 2 & 1 & 0 \\ -1 & 3 & 1 \end{pmatrix}_3 \times {}_3\begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 3 \\ 0 & 0 & 5 \end{pmatrix}_3 = {}_2\begin{pmatrix} 1 & - & - \\ - & - & - \end{pmatrix}_3$$

Multiplication

Properties: $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$

$\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$

$$\begin{cases} \mathbf{D} = \text{diag}\{d_1, d_2, \ldots, d_n\} \\ \mathbf{A} = [a_{ij}]_{n\times m} \end{cases}$$

$$\Rightarrow \mathbf{D}\times\mathbf{A} = \begin{bmatrix} d_1a_{11} & d_1a_{12} & \cdots & d_1a_{1n} \\ d_2a_{21} & d_2a_{22} & \cdots & d_2a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_na_{n1} & d_na_{n2} & \cdots & d_na_{nn} \end{bmatrix}$$

Exercise

Let $\mathbf{Y} = [y_{ij}]_{n\times p}$, where y_{ij} is the phenotypic score for trait j (j = 1,...,p) on animal i (i = 1,...,n).

Given the vectors of p means and p standard deviations relative to each trait, use matrix operations to obtain a matrix $\mathbf{Z} = [z_{ij}]_{n\times p}$ of standardized values: $z_{ij} = (y_{ij} - \mu_j)/\sigma_j$

$$\mathbf{Y}_{(n\times p)} = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1p} \\ y_{21} & y_{22} & \cdots & y_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{np} \end{bmatrix} \quad \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{bmatrix} \quad \sigma = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_p \end{bmatrix}$$

Exercise

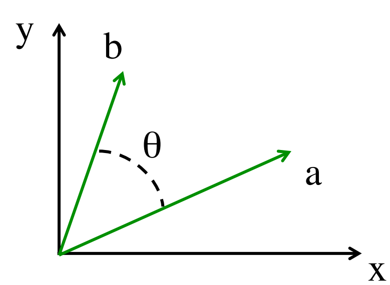
$$\mathbf{M}_{(n\times p)} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} \mu_1 & \mu_2 & \cdots & \mu_p \end{bmatrix} = \begin{bmatrix} \mu_1 & \mu_2 & \cdots & \mu_p \\ \mu_1 & \mu_2 & \cdots & \mu_p \\ \vdots & \vdots & \ddots & \vdots \\ \mu_1 & \mu_2 & \cdots & \mu_p \end{bmatrix}$$

$$\mathbf{S}_{(p\times p)} = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_p \end{bmatrix} \rightarrow \mathbf{P}_{(p\times p)} = \begin{bmatrix} 1/\sigma_1 & 0 & \cdots & 0 \\ 0 & 1/\sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/\sigma_p \end{bmatrix}$$

$$\mathbf{Z} = (\mathbf{Y} - \mathbf{M}) \times \mathbf{P}$$

Inner Product

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix} \rightarrow \mathbf{a}^T\mathbf{b} = \mathbf{b}^T\mathbf{a} = \sum_{i=1}^{k} a_i b_i = \langle \mathbf{a}, \mathbf{b} \rangle$$



Geometric interpretation:

$$\|\mathbf{a}\| = (\mathbf{a}^T\mathbf{a})^{1/2} = \sqrt{\sum_{i=1}^{k} a_i^2}$$

$$\|\mathbf{b}\| = (\mathbf{b}^T\mathbf{b})^{1/2} = \sqrt{\sum_{i=1}^{k} b_i^2}$$

Euclidean distance (norm)

$$\cos(\theta) = \frac{\mathbf{a}^T\mathbf{b}}{(\mathbf{a}^T\mathbf{a})^{1/2}(\mathbf{b}^T\mathbf{b})^{1/2}}$$

$\mathbf{a}^T\mathbf{b} = 0 \rightarrow$ a and b orthogonal

Direct (Kronecker) Product

$$A_{(m\times n)} \otimes B_{(p\times q)} = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}$$

Properties:

$$A \otimes (B+C) = A \otimes B + A \otimes C$$
$$(A \otimes B) \otimes C = A \otimes (B \otimes C)$$
$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$$
$$(A \otimes B)^{T} = A^{T} \otimes B^{T}$$
$$\mathrm{tr}(A \otimes B) = \mathrm{tr}(A)\mathrm{tr}(B)$$
$$\det(A \otimes B) = (\det A)^{m} \det(B)^{n}$$

Transpose

$$\mathbf{B} = \mathbf{A}^{T} \to b_{ij} = a_{ji}$$

Example:

$$\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 2 & -1 \\ 1 & 0 \end{pmatrix} \to \mathbf{A}^{T} = \begin{pmatrix} 3 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}$$

Properties:

$$(\mathbf{AB})^{T} = \mathbf{B}^{T}\mathbf{A}^{T}$$
$$(\mathbf{ABC})^{T} = \mathbf{C}^{T}\mathbf{B}^{T}\mathbf{A}^{T}$$

Sample Mean Vector and Co-Variance Matrix

$$\bar{\mathbf{Y}}^{(j)} = \frac{1}{n}\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}\begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1p} \\ y_{21} & y_{22} & \cdots & y_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{np} \end{bmatrix}$$

$$= \begin{bmatrix} \bar{y}_{\bullet 1} & \bar{y}_{\bullet 2} & \cdots & \bar{y}_{\bullet p} \end{bmatrix}$$ (trait means)

$$\mathbf{C} = \mathbf{Y} - \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}\begin{bmatrix} \bar{y}_1 & \bar{y}_2 & \cdots & \bar{y}_p \end{bmatrix} = \begin{bmatrix} y_{11} - \bar{y}_{\bullet 1} & \cdots & y_{1p} - \bar{y}_{\bullet p} \\ y_{21} - \bar{y}_{\bullet 1} & \cdots & y_{2p} - \bar{y}_{\bullet p} \\ \vdots & \ddots & \vdots \\ y_{n1} - \bar{y}_{\bullet 1} & \cdots & y_{np} - \bar{y}_{\bullet p} \end{bmatrix}$$

Sample Mean Vector and Co-Variance Matrix

$$\mathbf{C} = \mathbf{Y} - \frac{1}{n}(\mathbf{1}\times\mathbf{1}^T)\mathbf{Y} = \left(\mathbf{I} - \frac{1}{n}\mathbf{J}\right)\times\mathbf{Y}$$

$$\mathbf{C}^T\mathbf{C} = \begin{bmatrix} \sum_{i=1}^{n}(y_{i1}-\bar{y}_1)^2 & \sum_{i=1}^{n}(y_{i1}-\bar{y}_1)(y_{i2}-\bar{y}_2) & \cdots & \sum_{i=1}^{n}(y_{i1}-\bar{y}_1)(y_{ip}-\bar{y}_p) \\ \sum_{i=1}^{n}(y_{i1}-\bar{y}_1)(y_{i2}-\bar{y}_2) & \sum_{i=1}^{n}(y_{i2}-\bar{y}_2)^2 & \cdots & \sum_{i=1}^{n}(y_{i2}-\bar{y}_2)(y_{ip}-\bar{y}_p) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{n}(y_{i1}-\bar{y}_1)(y_{ip}-\bar{y}_p) & \sum_{i=1}^{n}(y_{i2}-\bar{y}_2)(y_{ip}-\bar{y}_p) & \cdots & \sum_{i=1}^{n}(y_{ip}-\bar{y}_p)^2 \end{bmatrix}$$

$$\mathbf{S} = \frac{1}{(n-1)}\mathbf{C}^T\mathbf{C}$$

Trace

- Trace: the trace of a square matrix A is simply the sum of its diagonal elements

$$\mathbf{A}_{(p\times p)} \rightarrow \mathrm{tr}(\mathbf{A}) = \sum_{k=1}^{p} a_{kk}$$

Properties: $\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA})$

$\mathrm{tr}(\mathbf{A}) = \mathrm{tr}(\mathbf{A}^T)$

Example: $\mathbf{A} = \begin{bmatrix} 1 & 0 & 4 \\ 3 & -3 & 2 \\ \sqrt{2} & -1 & 1 \end{bmatrix} \rightarrow \mathrm{tr}(\mathbf{A}) = 1 - 3 + 1 = -1$

Quadratic Forms

$$Q(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x} = \sum_{i\leq j} a_{ij}x_i x_j$$

➔ $\mathbf{A} = [a_{ij}]_{(n\times n)}$ is symmetric and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

Examples: $\mathbf{A} = \mathbf{I} \rightarrow Q(\mathbf{x}) = \sum_i x_i^2$

$\mathbf{A} = \mathbf{J} \rightarrow Q(\mathbf{x}) = \left(\sum_i x_i\right)^2$

Determinant

- 2 x 2 matrix:

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \det(\mathbf{A}) \text{ or } |\mathbf{A}| = ad - bc$$

- 3 x 3 matrix:

$$|\mathbf{B}| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= aei + bfg + cdh - ceg - bdi - afh$$ (Sarrus)

Determinant

- Any dimension (Laplace):

$$|\mathbf{A}| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} |\mathbf{A}_{ij}|$$ (for a fixed row i)

$$|\mathbf{A}| = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} |\mathbf{A}_{ij}|$$ (for a fixed column j)

Minor (see next slide)

Properties: $|c\mathbf{A}| = c|\mathbf{A}|$

$$\mathbf{D} = \mathrm{diag}(d_i) \rightarrow |\mathbf{D}| = \prod_{i=1}^{n} d_i$$

$$|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$$

Minor and Cofactor

- The (i,j) **minor** of a matrix is the determinant of the submatrix formed by deleting its i-th row and j-th column

$$\mathbf{A} = \begin{bmatrix} 3 & 4 & -1 \\ 0 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \rightarrow \mathbf{A}_{2,3} = \det \begin{bmatrix} 3 & 4 & \square \\ \square & \square & \blacksquare \\ 1 & 2 & \square \end{bmatrix} = \det \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} = 2$$

- The (i,j) **cofactor** of a matrix is obtained by multiplying its corresponding minor by $(-1)^{i+j}$, i.e. $\mathbf{C}_{ij} = (-1)^{i+1}\mathbf{A}_{ij}$

For the example above: $\mathbf{C}_{23} = (-1)^{2+3}\mathbf{A}_{2,3} = (-1) \times 2 = -2$

Inverse

Square matrix: $\mathbf{A}$

- Inverse: $A^{-1} \rightarrow A^{-1}A = AA^{-1} = \mathbf{I}$

Example 1: $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \rightarrow \mathbf{A}^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \times \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{cases} a + 2c = 1 \\ -a + 0c = 0 \\ b + 2d = 0 \\ -b + 0d = 1 \end{cases} \Rightarrow \begin{cases} a = 0 \\ b = -1 \\ c = 0.5 \\ d = 0.5 \end{cases} \rightarrow \mathbf{A}^{-1} = \begin{pmatrix} 0 & -1 \\ 0.5 & 0.5 \end{pmatrix}$$

Inverse

- Inverse of a 2 x 2 matrix: $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$\mathbf{A}^{-1} = \frac{1}{(a_{11}a_{22} - a_{12}a_{21})}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

$\det(\mathbf{A})$

Example: $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \rightarrow \mathbf{A}^{-1} = \frac{1}{2}\begin{pmatrix} 0 & -2 \\ 1 & 1 \end{pmatrix}$

Question: What if det(A) = 0 ?

Inverse

- Higher dimension matrices: inverse of the adjugate matrix times the reciprocal of the determinant

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\operatorname{adj}(\mathbf{A})$$

Adjugate (or adjunct) of A: $\operatorname{adj}(\mathbf{A}) = \mathbf{C}^{T}$

Cofactor of A: matrix formed by all of the cofactors of a square matrix

If $\det(\mathbf{A}) = 0 \rightarrow \mathbf{A}^{-1}$ does not exist
(A is said to be singular)

Example 2: $\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \rightarrow \mathbf{A}^{-1} = ?$

$$\det(\mathbf{A}) = 1 + 0 + 0 - 0 - 0 - 2 = -1$$

$$\mathbf{C} = \begin{pmatrix} 1 & -1 & -1 \\ -2 & 1 & 3 \\ 0 & 0 & -1 \end{pmatrix} \rightarrow \text{adj}(\mathbf{A}) = \begin{pmatrix} 1 & -2 & 0 \\ -1 & 1 & 0 \\ -1 & 3 & -1 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \begin{pmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ 1 & -3 & 1 \end{pmatrix}$$

Inverse

- Properties and useful identities:

$$\mathbf{A} \text{ symmetric} \rightarrow \mathbf{A}^{-1} \text{ symmetric}$$

$$(\mathbf{A}^{\mathrm{T}})^{-1} = (\mathbf{A}^{-1})^{\mathrm{T}}$$

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

$$(\mathrm{c}\mathbf{A})^{-1} = \mathrm{c}^{-1}\mathbf{A}^{-1} \text{ (c is a scalar)}$$

$$\mathbf{D} = \text{diag}(\mathrm{d}_i) \rightarrow \mathbf{D}^{-1} = \text{diag}(1/\mathrm{d}_i)$$

Rank

- The rank of a matrix is defined as the maximum number of linearly independent column vectors (or row vectors) in the matrix

➔ Let $\mathbf{A}_{(m \times n)}$ with $\text{rank}(\mathbf{A}) = k$

1. If $m = n = k \rightarrow \mathbf{A}$ is of full rank and so it is non-singular, i.e. $\exists \mathbf{A}^{-1}$
2. If $m = n > k \rightarrow \mathbf{A}$ is of incomplete rank and so it is singular, i.e. $\nexists \mathbf{A}^{-1}$
3. If $m \neq n$ then $\mathbf{A}$ is not square and it does not make sense to talk about $\mathbf{A}^{-1}$

Generalized Inverse

- Generalized inverse: $A^{-} \rightarrow AA^{-}A = A$

➔ Examples: Moore-Penrose, conditional, least squares, reflexive,...

➔ Searle (1971) algorithm for conditional:

i) Given a matrix $A_{(m \times n)}$, with rank(A) = k, find a non-singular submatrix $M_{(k \times k)}$ and compute $(M^{-1})^T$

ii) Substitute the elements of M in A for $(M^{-1})^T$, and fill with zeros the remaining elements

iii) The transpose of this matrix is a generalizd inverse of A

Example

matrix operations

$$\mathbf{A} = \begin{bmatrix} 2 & 2 & 0 \\ 4 & 2 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \text{rank}(\mathbf{A}) = 2$$

$$\mathbf{M} = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix} \rightarrow \left(\mathbf{M}^{-1}\right)^{T} = \begin{bmatrix} 0.5 & 0 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{A}^{-} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.5 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$AA^{-}A = A$$

Linear System of Equations

$$\begin{cases} x_1 + 2x_2 - x_3 = 4 \\ 2x_1 - 2x_2 + x_3 = 2 \\ x_1 - x_2 - x_3 = 1 \end{cases}$$

- Elimination, Substitution, etc.

$$\begin{array}{r} x_1 + 2x_2 - x_3 = 4 \\ 2x_1 - 2x_2 + x_3 = 2 \\ \hline 3x_1 + 0 + 0 = 6 \end{array} \; +$$

$$x_1 = 2$$

$$\begin{array}{r} -2x_2 + x_3 = -2 \\ -x_2 - x_3 = -1 \\ \hline -3x_2 + 0 = -3 \end{array} \; +$$

$$x_2 = 1$$
$$x_3 = 0$$

Linear System of Equations

- Using matrices:

$$\underbrace{\begin{pmatrix} 1 & 2 & -1 \\ 2 & -2 & 1 \\ 1 & -1 & -1 \end{pmatrix}}_{C}\underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}}_{r}$$

system of equations

$$Cx = r \rightarrow x = C^{-1}r$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 & 0 \\ 1/3 & 0 & -1/3 \\ 0 & 1/3 & -2/3 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Linear System of Equations

1) Single solution
2) Infinitely many solutions: consistent
3) No solutions: inconsistent

Note:
More equations than variables: overdetermined system
More variables than equations: underdetermined system

Eigenvalues and Eigenvectors

- Characteristic equation: $|\mathbf{C} - \lambda\mathbf{I}| = 0$

  $\mathbf{C}_{(nxn)}$ symmetric

- Solutions to λ are an n-degree polynomial
- The n λ's that are the roots of this polynomial are the eigenvalues of $\mathbf{C}$
- For each λ, one can define a nonzero vector $\mathbf{a}$, such that: $(\mathbf{C} - \lambda\mathbf{I})\mathbf{a} = \mathbf{0}$
- Any $\mathbf{a}$ that satisfies this equation is called a latent vector or eigenvector of $\mathbf{C}$. Each eigenvector is associated with its own λ.

Eigenvalues and Eigenvectors

- Example: Correlation matrix $\mathbf{C} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$

$$\mathbf{C} - \lambda\mathbf{I} = \begin{bmatrix} 1-\lambda & \rho \\ \rho & 1-\lambda \end{bmatrix} \rightarrow |\mathbf{C} - \lambda\mathbf{I}| = (1-\lambda)^2 - \rho^2$$

$$(1-\lambda)^2 - \rho^2 = 0 \rightarrow \lambda = 1 \pm \rho$$

➔ Eigenvalues: $\lambda_1 = 1 + \rho$ and $\lambda_2 = 1 - \rho$

Eigenvalues and Eigenvectors

$$(\mathbf{C}-\lambda\mathbf{I})\mathbf{a} = \begin{bmatrix} 1-\lambda & \rho \\ \rho & 1-\lambda \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} (1-\lambda)a_1 + \rho a_2 \\ \rho a_1 + (1-\lambda)a_2 \end{bmatrix}$$

Equating it to zero: $\begin{cases} (1-\lambda)a_1 + \rho a_2 = 0 \\ \rho a_1 + (1-\lambda)a_2 = 0 \end{cases}$

➔ $\lambda_1 = 1+\rho \rightarrow \rho a_1 - \rho a_2 = 0 \rightarrow a_1 = a_2$

➔ $\lambda_2 = 1-\rho \rightarrow \rho(a_1 + a_2) = 0 \rightarrow a_1 = -a_2$

Cholesky Decomposition

➔ If ***A*** is a positive definite matrix, with real-valued entries:

$$\boldsymbol{A} = \mathbf{L}\mathbf{L}^T$$

where **L** is a lower triangular matrix

➔ Useful factorization for efficient numerical solutions and Monte Carlo simulations